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Existence and Uniqueness Theory of the Vlasov–Poisson System with Application to the Problem with Cylindrical Symmetry

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1. INTRODUCTION

The Vlasov–Poisson system is a nonlinear integro-differential system of equations which describes the motion of charged particles in the presence of the electrostatic force field which the particles themselves generate. The system can be written for any number of interacting positively and negatively charged species. For simplicity the system that is given here is for a single charged species, say electrons. Also, the system is commonly written for a one-, two-, or three-dimensional Euclidean space. In this paper the space is three dimensional. The Vlasov–Poisson system that we study is the following:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_v f = 0, \tag{a}$$

$$f(x, v, 0) = g(x, v), \tag{1.1}$$

$$\Delta \Phi = -4\pi^2 \int f\, dv, \tag{b}$$

$$\nabla_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_3}\right), \qquad \nabla_v = \left(\frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial v_3}\right),$$

$$\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2},$$

$x = (x_1, \ldots, x_3)$ a point in position space,

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$v = (v_1, ..., v_3)$ a point in velocity space.

(x, v) a point in 6-dimensional phase space.

The function f represents number density of electrons in phase space. The function Φ is the electrostatic potential generated by f. The analysis presented in this paper is for the system (1.1a, b); however, more complicated systems involving several different charged species can be handled by the same techniques. Also, if the sign is changed in front of $\nabla_x \Phi$ in the transport equation, (1.1a, b) is then the system of stellar dynamics. Our analysis applies to the system in this form as well.

For the dimension of the space less than three, global-in-time existence and uniqueness of classical solutions to the Vlasov–Poisson system has been proved. Proofs for the one-dimensional system are in [8, 11]. Proofs for two dimensions are given in [10, 12]. For the three-dimensional system, (1.1a, b), local-in-time proofs for the existence and uniqueness of classical solutions are given in [1, 2, 4, 9–11]. The problem of global-in-time solvability of (1.1a, b) for a general class of initial data is still open. However, for certain classes of symmetric initial data proofs of global-in-time existence and uniqueness of classical solutions have been given. For spherically symmetric initial data a proof for the stellar dynamic problem is carried out in [2]. In [13] this result is generalized to include the electrostatic problem as well. Recently the cylindrically symmetric problem has been solved by Horst in his thesis [4]. Those results are refined in [3, 5–7].

The purpose of this paper is first to present some general theoretical results on existence and uniqueness of classical solutions to (1.1a, b) and second to apply the general theory to give another proof for the cylindrically symmetric problem. In so doing we give another description of the cylindrically symmetric problem, gain a better understanding of the role the symmetry plays in the solution of the problem, and make additional refinements to the solution. Section 2 is devoted to general theory. A theorem is stated and proved which gives a condition for solvability of system (1.1a, b). Similar theorems are proved in [3, 5, 6]; however, there are some significant differences between these results and the ones obtained here. A reduction of the system under cylindrical symmetry is carried out in Section 3. In Section 4 expressions are obtained for the r, z components of the symmetric field from which various estimates on the field are derived. We see clearly here how types of estimates which appear inadequate for a proof in the general case reduce under symmetry to estimates which are sufficient for the symmetric case. In Section 5 two theorems are proved on global existence and uniqueness of solutions to the cylindrically symmetric system. This proofs are modeled after the type of result obtained in [13] for the spherically symmetric problem. The proofs depend on getting a priori bounds on the growth of velocities within the system. In the first case we prove a result for a system in which angular momentum is bounded away from zero. We can get better estimate on velocities for this case. In the second theorem restrictions on angular momentum are removed and the more general symmetric problem is solved. The proofs give a rather different characterization of the cylindrically symmetric system from that given in [3–7]. The a priori bounds on velocities are more precise than in these previous works.[1]

2. Notation: General Theory

Notation

The following notation is used throughout:

R_n: n-dimensional Euclidean space, n-dimensional phase space, linear space of n-tuples (depending on the context).

For a set $A \subseteq R_n$:

$C(A)$: the class of continuous, bounded functions defined on the domain A.

$C'(A)$: the subset of $C(A)$ comprised of functions having bounded, continuous first derivatives on A.

If A is unbounded:

$C_0(A)$, $C_0'(A)$: the subsets of $C(A)$, $C'(A)$ containing functions whose support in A is bounded.

For the function f on A:

$\sup_A |f|$: supremum of f on A.

$\|f\|_p$: L_p norm of f (on A)

$\operatorname{supp} f$: support of f (in A).

[1] An earlier version of this paper had a proof only for the case with angular momentum bounded away from zero. The result in [4] is also of this type. Papers [3, 5, 7] have proofs with no restriction on angular momentum. This was a motivation for making a similar extension of the present results which is done in the second theorem of Section 5. Also, the estimates on the field in Section 4 are improvements over similar types of estimates in [3, 4]. Some of these same estimates are obtained in [5, 7] as well.

For a point $x = (x_1, x_2, x_3) \in R_3$:

$$|x| = \sum_{i=1}^{3} |x_i|.$$

Other notation will be introduced as it is needed.

General Theorem: Integral Bounds and Conservation Laws

The treatment of classical existence theory given here is for initial data of class $C_0'(R_6)$. This is a natural class of initial data to consider in the sense that real physical distributions have finite extent and velocity. Also, for this class of data we can get the simplest formulation of a general theory of the problem. In [2, 6, 10], theories are developed which deal with some more general classes of initial data.

If the initial data g to (1.1a, b) is of class $C_0'(R_6)$, then it is known that the classical solution to (1.1a, b) exists and is unique on at least some time interval $[0, \alpha]$, α a positive number. A proof of this was first given by Kurth [9]. The question is what happens as α gets large? Either the classical solution continues to exist for all time or a finite positive number T exists such that the solution *blows up* in some fashion as $t \to T$. Since the solution is a constant along characteristics, then the supremum norm of the solution does not *blow up*. Thus if the classical solution ceases to exist at T it must be because a derivative becomes unbounded or the support becomes unbounded as $t \to T$. In [2], however, it is shown that as long as the support of the solution remains bounded in velocity space, then the derivatives remain bounded. Clearly, support in position space also remains bounded. It is thus evident that for initial data of class $C_0'(R_6)$ the solution to (1.1a, b) continues to exist as a classical solution to the problem as long as the support of the solution remains bounded. Furthermore, as we shall see the classical solution ceases to exist if the support of the solution becomes unbounded. We make these comments precise by stating Theorem 2.1. The proof will be given later in this section. Similar types of results are obtained in [3, 5, 6].

THEOREM 2.1. *Let g be a function in $C_0'(R_6)$. The solution to* (1.1a, b) *exists and is unique as an element of $C'(R_6 \times [0, T])$ if and only if the system* (1.1a, b) *admits an a priori bound on the support of C' solutions for t in the interval* $[0, T]$.

To prove Theorem 2.1 and to apply the result we need to get estimates for the field $\nabla_x \phi$. To get estimates we rely on certain integral conservation laws which the solution to system (1.1a, b) obeys and certain integral bounds which are consequences of these laws. These various conservation laws and bounds are stated in the lemmas that follow. We assume for the initial data g that

$$\sup |g| = M_0, \tag{2.2i}$$

$$\|g\|_1 = M, \tag{2.2ii}$$

$$\|g\|_p = M(p) \qquad 1 < p < \infty, \tag{2.2iii}$$

$$\operatorname{supp} g \subseteq \{(x, v)/|x| \leqslant R_0, |v| \leqslant R_0\}. \tag{2.2iv}$$

LEMMA 2.1 (Conservation of mass). *If f is a solution to* (1.1a, b) *with initial data g, then $\|f\|_1 = \|g\|_1 = M$.*

Proof. Integrating (1.1a) over x, v we get

$$\frac{\partial}{\partial t} \int f \, dv \, dx = 0.$$

The assumption will be made that $g \geqslant 0$ and hence $f \geqslant 0$, thus

$$\|f\|_1 = \int f = \int g = \|g\|_1.$$

A generalization of Lemma 2.1 is

LEMMA 2.2. *If f is a solution to* (1.1a,b) *with initial data g, then $\|f\|_p = \|g\|_p = M(p)$.*

Proof. Multiplying system (1.1a) by f^{p-1} and integrating gives

$$\frac{\partial}{\partial t} \int f^p = 0.$$

The proofs of Lemmas 2.1 and 2.2 given here are simplified by the fact that $g \geqslant 0$ and hence $f \geqslant 0$. This condition is satisfied for all physical problems; however, the lemmas are still valid even if g and hence f take on negative values. (See [4, p. 27]).

LEMMA 2.3 (Conservation of energy). *Let f be a solution to* (1.1a,b), *Φ the potential which satisfies* (1.1b), *then*

$$\int \sum |v_i|^2 f \, dv \, dx + \int \sum |\Phi x_i|^2 \, dx = E; \tag{2.3}$$

i.e., total energy (kinetic + potential) is conserved.

Proof. Left to reader.

A further conservation law obeyed by system (1.1a,b) is the conservation of momentum which states

$$\int v_i f\, dv\, dx = c_i, \qquad i = 1, 2, 3.$$

We shall not, however, make use of these integrals, in getting our estimates.

On the basis of Lemmas 2.1–2.3 we can derive some further useful integral bounds for the solution to (1.1a,b). The next estimate (given in Lemma 2.4) is particularly important and is proved by Horst in his thesis.

LEMMA 2.4 (Horst). *If f is a solution to* (1.1a,b) *then*

$$\left(\int \left(\int f\, dv\right)^s dx\right)^{1/s} \leqslant K \qquad \text{for} \quad 1 \leqslant s \leqslant 5/3, \quad K = \text{const.}$$

Proof. The proof is given in [4, p. 42] and is repeated here. Since both terms in (2.3) are positive (electrostatic case) it follows that

$$\int_{x,v} \sum |v_i|^2 f \leqslant E.$$

That the kinetic energy is bounded can also be proved in general (i.e., for the gravitational problem as well, see [4, p. 44]). Let

$$L(p) = \left(\int_v f^p\right)^{1/p}, \qquad J = \int_v \sum |v_i|^2 f,$$

$$\int_v f \leqslant \int_{|v| \leqslant R} f + \frac{1}{R^2} \int \sum |v_i|^2 f \leqslant R^{3/q} L(p) + \frac{1}{R^2} J \quad (1/q + 1/p = 1).$$

Setting $R = (J/L(p))^{q/(3+2q)}$,

$$\int_v f \leqslant L(p)^{(2q/(3+2q))} J^{3/(3+2q)} = \left(\int f^p\right)^{2/(5p-3)} J^{((3p-3)/(5p-3))}.$$

Given $p \in [1, \infty]$ let

$$s = (5p-3)/(3p-1).$$

Then $s \in [1, 5/3]$ and

$$2s/(5p-3) + (3p-3)\, s/(5p-3) = 1.$$

It follows that

$$\left(\int (f)^s\right)^{1/s} \leqslant \left(\int \int f^p\right)^{2/(5p-3)} \left(\int_{x,v} \sum |v_i|^2 f\right)^{(3p-3)/(5p-3)}$$
$$\leqslant (\|f\|_p)^{2p/(5p-3)} \left(\int \sum |v_i|^2 f\right)^{((3p-3/(5p-3))}$$
$$\leqslant (M(p))^{2p/(5p-3)}\, E^{(3p-3)/(5p-3)}, \qquad 1 \leqslant p \leqslant \infty.$$

Then

$$\left(\int (\textstyle\int f)^s\right)^{1/s} \leqslant K, \qquad 1 \leqslant s \leqslant 5/3.$$

It is this bound given in Lemma 2.4 which allows the proof for the cylindrically symmetric case to go through. Previous proofs [2, 10, 11] utilize conservation of mass (Lemma 2.1) in obtaining estimates for the potential. Through the bound in Lemma 2.4 we also introduce conservation of energy into the estimates.

A final integral bound also a consequence of the conservation of energy is given in the next lemma. A bound of this type is used in [3].

LEMMA 2.5. *Assuming $g \in C_0'(R_6)$ the constant C exists such that the solution $f(\cdot, t)$ to* (1.1a, b) *has the bound*

$$\int \sum |x_i|^2 f \leqslant C(1+t)^2.$$

Proof.

$$\frac{\partial}{\partial t}\int_{x,v} x_i^2 f \cdot + \int_{x,v} x_i^2 \frac{\partial}{\partial x_i} v_i f = 0, \qquad \frac{\partial}{\partial t}\int x_i^2 f = 2\int x_i v_i f.$$

Hence,

$$\int x_i^2 f \leqslant \int x_i^2 g + 2\int_0^t \left(\int x_i^2 f\right)^{1/2} \left(\int v_i^2 f\right)^{1/2} d\bar{t}$$
$$\leqslant \int x_i^2 g + 2E^{1/2} \int_0^t \left(\int x_i^2 f\right)^{1/2} d\bar{t}.$$

Let $C > 3\max(\int x_i^2 g, 2E^{1/2})$, $i = 1, 2, 3$. It follows that

$$\int \sum |x_i|^2 f \leqslant C(1+t)^2.$$

Bounds on the Potential Function

Estimates for the field $\nabla_x \Phi$ are obtained using the integral bounds from Lemmas 2.1–2.4. These bounds are known a priori from the initial data. Let

$$h(x, t) = \int f \, dv$$

and let

$$A = \sup_{\Omega} |h|, \qquad D = \sup_{\Omega} |h_{x_i}|, \qquad i = 1, 2, 3, \quad \Omega = R_3 \times [0, T].$$

The first estimate which we compute is in terms of $\sup |h| = A$ and is obtained in [3]. From Lemma 2.4 we have that

$$\left(\int h^{5/3}\right)^{3/5} \leqslant K,$$

$$\frac{\partial \Phi}{\partial x_i} = \int \frac{h(x_i - \bar{x}_i)}{r^3} \, d\bar{x}$$

$$= \int_{r \leqslant \varepsilon} \frac{h(x_i - \bar{x}_i)}{r^3} \, d\bar{x} + \int_{r \geqslant \varepsilon} \frac{h(x_i - \bar{x}_i)}{r^3} \, d\bar{x},$$

$$r = ((x_1 - \bar{x}_1)^2 + \cdots + (x_3 - \bar{x}_3)^2)^{1/2},$$

$$\left| \int_{r \leqslant \varepsilon} \frac{h(x_i - \bar{x}_i)}{r^3} \, d\bar{x} \right| \leqslant A \int_0^{\varepsilon} \frac{\bar{r}^2 \, d\bar{r} \sin \phi \, d\phi \, d\theta}{\bar{r}^2} \leqslant 2\pi A \varepsilon,$$

$$\left| \int_{r > \varepsilon} \frac{h(x_i - \bar{x}_i)}{r^3} \right| \leqslant \left(\int h^{5/3}\right)^{3/5} \left(\int_{r \geqslant \varepsilon} \left(\frac{1}{r^2}\right)^{5/2} d\bar{x}\right)^{2/5}$$

$$\leqslant K\pi^{2/5}/\varepsilon^{4/5}.$$

Thus

$$\left| \frac{\partial \Phi}{\partial x_i} \right| \leqslant 2\pi A \varepsilon + K\pi^{2/5}/\varepsilon^{4/5}.$$

Letting $\varepsilon = 1/A^{5/9}$ we get an estimate

$$\left| \frac{\partial \Phi}{\partial x_i} \right| \leqslant BA^{4/9}; \qquad B\text{-constant which depends on } K. \tag{2.4}$$

The next estimate is in terms of $\sup |h_{x_i}| = D$ and is used in the proof of Theorem 2.1.

$$\frac{\partial \Phi}{\partial x_i} = \int \frac{h(x_i - \bar{x}_i)}{r^3}\, d\bar{x}$$

$$= \int_{r \leqslant \varepsilon} h_{\bar{x}_i} \frac{1}{r}\, d\bar{x} - \int_{r=\varepsilon} \frac{hn_i\, ds}{r} + \int_{r \geqslant \varepsilon} \frac{h(x_i - \bar{x}_i)}{r^3}\, d\bar{x},$$

where ds is an element of surface area on the sphere of radius ε and n_i is the projection of the unit normal to the surface on the $\bar{x}_i$ axis. Since

$$\int_{r=\varepsilon} \frac{h(x)\, n_i\, ds}{r} = h(x) \int_{r=\varepsilon} \frac{n_i\, ds}{r} = 0$$

we can write

$$\left| \int_{r=\varepsilon} \frac{h(\bar{x})\, n_i\, ds}{r} \right| = \left| \int_{r=\varepsilon} \frac{(h(\bar{x}) - h(x)\, n_i\, ds}{r} \right| \leqslant \int \frac{Dr\, |n_i|\, ds}{r}$$

$$\leqslant D \int_{r=\varepsilon} r^2 \sin \phi\, d\phi\, d\theta = 4\pi D \varepsilon^2$$

$$\int_{r \leqslant \varepsilon} |h_{\bar{x}_i}| \frac{1}{r}\, dx \leqslant 4\pi D \int_{r \leqslant \varepsilon} \frac{1}{r} r^2\, dr = 2\pi D \varepsilon^2$$

$$\int_{r \geqslant \varepsilon} \frac{|h|\, |x_i - \bar{x}_i|}{r^3}\, dx \leqslant K\pi^{2/5}/\varepsilon^{4/5}$$

(as in the computation of estimate 2.4).

Thus

$$\left| \frac{\partial \Phi}{\partial x_i} \right| \leqslant 6\pi D \varepsilon^2 + K\pi^{2/5}/\varepsilon^{4/5}.$$

Let $\varepsilon = 1/D^{5/14}$. We get an estimate

$$\left| \frac{\partial \Phi}{\partial x_i} \right| \leqslant BD^{2/7}; \qquad B = \text{const depending on } K. \tag{2.5}$$

Proof of Theorem; Local Existence and uniqueness.

We now give the proof of Theorem 2.1.

Proof (*Theorem* 2.1). We first prove *sufficiency* by assuming the constant R exists such that if f is a classical solution to (1.1a, b) for $t \in [0, T]$ then

$$\operatorname{supp} f(\cdot, t) \subset \{(x, v), |x| \leqslant R, |v| \leqslant R\}.$$

In [2, Lemma 2, p. 351] it is proved that given a bound on support a bound on the derivatives of f follows. The result in [2] is given, however, in terms of certain invariant subspaces rather than in terms of *a priori bounds*. A version of the result more applicable to the development of theory given here is derived in [12] for the two-dimensional problem. To get the same result in three dimensions we can use the bounds on the potential function given in [2, pp. 344, 345] and carry out the computation in [12] in three dimensions. We obtain thereby an a priori bound on the derivatives of f for $t \in [0, T]$.

On the basis of the a priori bounds on the support and derivatives of a solution for $t \in [0, T]$, the classical solution to (1.1a, b) is then constructed as the limit in the continuous function norm of the sequence of functions $\langle f_n \rangle$ $n = 1, 2, \ldots,$ where f_n $n \geqslant 2$ is the solution to

$$\frac{\partial f_n}{\partial t} + v \cdot \nabla x f_n + \nabla x \Phi_{n-1} \cdot \nabla v f_n = 0 \tag{2.6a}$$

$$\Delta \Phi_{n-1} = -4\pi^2 \int_v f_{n-1}\, dv, \qquad f_n(0) = g \qquad \text{and} \qquad f_1 = g. \tag{2.6b}$$

The proof of convergence is well known and will not be presented. For details the reader is referred to [2, 11, 12]. In [12] proofs are given in two space dimensions, but at this point the proof is intrinsically the same in either two or three dimensions.

To prove *necessity* one assumes the function f of class $C'(R_6 \times [0, T])$ is a solution to (1.1a, b). It is necessary to show that f has compact support.

Let

$$\sup_{\Omega} |f_{x_i}| = L \qquad i = 1, 2, 3, \quad \Omega = R_6 \times [0, T].$$

The bound on support is obtained by solving the characteristic equations

$$\frac{dx_i}{dt} = v_i, \tag{2.7a}$$

$$\frac{dv_i}{dt} = \frac{\partial \Phi}{\partial x_i}, \qquad i = 1, 2, 3. \tag{2.7b}$$

For $(x, v) \in R_6$ the function $P = P(x, v)$ is defined as

$$P(x, v) = \sum_{i=1}^{3} |v_i| = |v|.$$

Let $A(t) = \operatorname{supp} f(\cdot, t)$, the support of f in R_6 at time t and

$$q(t) = \sup_{A(t)} P.$$

For $h_{x_i} = \int f_{x_i}\, dv$, then $\sup_{R_3} |h_{x_i}| \leqslant 8L(q(t))^3$. Assume Φ is the solution to (1.1b). It follows from estimate (2.5) that

$$\left| \frac{\partial \Phi}{\partial x_i} \right| \leqslant B(8L)^{2/7}\, (q(t))^{6/7}.$$

Utilizing this bound and integrating (2.7) we get that

$$|v_i(t)| \leqslant |v_i(0)| + B(8L)^{2/7} \int_0^t (q(s))^{6/7}\, ds.$$

Summing over i and taking the supremum over trajectories originating at points in the support g, it can be shown that the bound $q(t)$ satisfies

$$q(t) \leqslant 3R_0 + 3B(8L)^{2/7} \int_0^t (q(s))^{6/7}\, ds. \tag{2.8}$$

Hence,

$$\begin{aligned} q(t) &\leqslant ((3R_0)^{1/7} + (3B/7)(8L)^{2/7}\, t)^7 \\ &\leqslant ((3R_0)^{1/7} + (3B/7)(8L)^{2/7}\, T)^7 \qquad \text{for} \quad t \in [0, T]. \end{aligned} \tag{2.9}$$

This gives a bound for the support of f in velocity space. Integrating (2.7) and using (2.9) gives a bound for the support of f in position space. We thus have a bound on the support in R_6 of the solution f for each $t \in [0, T]$.

In [3, 5, 6] theorems similar in nature to Theorem 2.1 are proved. The "only if" parts of the theorems, however, are proved by assuming boundedness of the field or boundedness of support as a part of the definition of *classical solution*. In the present paper *classical solutions* is defined in the standard way as a C' function that solves the problem. Boundedness of the field (and hence support) is then shown to be a consequence of the definition.

As an elementary application of Theorem 2.1 we prove the following local-in-time existence and uniqueness theorem for system (1.1a, b).

THEOREM 2.2. *Let the initial data g of* (1.1a, b) *be of class* $C_0'(R_6)$ *and have the bounds* (2.2). *The classical solution to* (1.1a, b) *exists and is unique as an element of* $C'(R_6 \times [0, \alpha])$ *for*

$$\alpha < [(3R_0)^{1/3} B(8M_0)^{4/9}]^{-1},$$

where B is the constant in (2.4).

Proof. Let the variable $q(t)$ be defined as in the proof of Theorem 2.1. Let

$$h = \int f\,dv.$$

Then $\sup_{R_3} |h| \leqslant 8M_0(q(t))^3$. From estimate (2.4) we have that

$$\left| \frac{\partial \Phi}{\partial x_i} \right| \leqslant B(8M_0)^{4/9}\,(q(t))^{4/3}.$$

Integrating the characteristic system (2.7) we compute that $q(t)$ satisfies

$$\begin{aligned} q(t) &\leqslant q(0) + 3B(8M_0)^{4/9} \int_0^t (q(s))^{4/3} \\ &\leqslant 3R_0 + 3B(8M_0)^{4/9} \int_0^t (q(s))^{4/3}\,ds. \end{aligned}$$

Hence

$$q(t) \leqslant 3R_0/(1-(3R_0)^{1/3} B(8M_0)^{4/9}\,t)^3.$$

For $t \leqslant \alpha < [(3R_0)^{1/3} B(8M_0)^{4/9}]^{-1}$ $q(t)$ remains bounded. This gives an a priori bound for the support of f in velocity space (and upon integration in position space). From Theorem 2.1 it therefore follows that the existence and uniqueness of the classical solution is guaranteed for the interval of time $[0, \alpha]$ for α as bounded above. (Note that B in the bound for α depends on the constant K, the bound in Lemma (2.4).)

To get a global-in-time solution to (1.1a,b) we need to be able to integrate the characteristic system (2.7) to get a solution that is bounded in time. We get such a solution if we have a bound for the field of the type

$$\left| \frac{\partial \Phi}{\partial x_i} \right| \leqslant BA^{\delta}\ \delta \leqslant \frac{1}{3} \qquad \left(\text{instead of } \delta = \frac{4}{9}\right). \tag{2.10}$$

So far a bound of this type with a suitable exponent on A has not been found for the general problem. If cylindrical symmetry is introduced into the problem certain improvements can be made in the estimates for the potential. As will be seen for points off axis estimates of the type (2.10) with exponent $\delta \leqslant \frac{1}{3}$ are obtainable; the constant B, however, depends on r (or r and t) and $B \to \infty$ as $r \to 0$. With the refinements to the estimates of the potential due to symmetry we are able to compute a priori bounds for the support of solutions in the cylindrically symmetric case. The remainder of the paper is devoted to the cylindrically symmetric problem.

For the development that follows we make further use of the characteristic system (2.7). A solution to (2.7) is written

$$(x(t), v(t)) = (x_1(t), \ldots, v_3(t)).$$

This is a curve in R_6 parametrized by t and is also referred to as a trajectory of the system (1.1a,b).

3. Cylindrically Symmetric Vlasov–Poisson System

Let $z = x_3$ be the axis of symmetry in position space. A cylindrically symmetric solution to system (1.1a,b) is characterized by the two conditions

I. A trajectory $(x(t), v(t))$ of (1.1a,b) satisfies

$$x_1(t)\, v_2(t) - x_2(t)\, v_1(t) = a \qquad (a \text{ const}).$$

II. The solution of (1.1a,b) is a function of $r = (x_1^2 + x_2^2)^{1/2}$, $z = x_3$ and t alone independent of azimuthal angle θ.

A cylindrically symmetric solution can be written as a function of t and the set of five independent variables

$$r = (x_1^2 + x_2^2)^{1/2}, \qquad \eta = (x_1 v_1 + x_2 v_2),$$
$$\xi = (x_1 v_2 - x_2 v_1), \qquad z = x_3, \qquad v_z = v_3. \tag{3.1}$$

Let $s(x, v)\colon R_6 \to R_5$ be the vector valued function given by (3.1). Initial data of the form

$$g(x, v) = G(s(x, v)) \tag{3.2}$$

is termed cylindrically symmetric. It is easy to see that for data of the form (3.2) solutions to (1.1a,b) satisfy conditions I, II. The solution f is a function of the form

$$f(x, v, t) = F(s(x, v), t), \tag{3.3}$$

where the function F satisfies a system in the variables (3.1)

$$\frac{\partial F}{\partial t} + (\eta/r)\frac{\partial F}{\partial r} + \left((\eta/r)^2 + (\xi/r)^2 + r\frac{\partial \psi}{\partial r}\right)\frac{\partial F}{\partial \eta} + v_z\frac{\partial F}{\partial z} + \frac{\partial \psi}{\partial z}\frac{\partial F}{\partial v_z} = 0$$

$$F(\cdot, 0) = G \tag{3.4a}$$

$$\frac{\partial^2 \psi}{\partial r^2} + 1/r\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = -4\pi^2 \int F \, d\eta \, d\xi \, dv_z. \tag{3.4b}$$

The variable ξ gives at a point in phase space the component of angular momentum around the $r = 0$ axis. From condition (I) we have that $\xi = a$ (const) along a characteristic curve of (3.4). That is, along a trajectory the angular momentum is constant. Thus system (3.4) splits naturally into component equations for the regions $\xi < 0$, $\xi > 0$, and $\xi = 0$.

If we write

$$\begin{aligned} G_1 &= G, & \xi &> 0, \\ &= 0, & \xi &\leqslant 0. \\ G_2 &= G, & \xi &< 0, \\ &= 0, & \xi &\geqslant 0. \\ G_3 &= G, & \xi &= 0, \\ &= 0, & \xi &\neq 0. \end{aligned}$$

$$G = G_1 + G_2 + G_3.$$

Then the solution F to (3.4) is a function of the form

$$F = F_1 + F_2 + F_3,$$

where F_1 has support in $\xi > 0$, satisfies (3.4a) and has the value G_1 at $t = 0$. Now F_2 has support in $\xi < 0$ and satisfies (3.4a) with initial data G_2 at $t = 0$; F_3 satisfies the reduced equation for $\xi = 0$ and has initial value G_3 at $t = 0$. Here F_1 is a distribution for left or counter clockwise rotating matter, F_2 the distributions for right or clockwise rotating matter and F_3 is the distribution for matter moving in the planes containing the $r = 0$ axis. Thus a cylindrically symmetric solution is made up of these three separate distributions which remain distinct in that particles do not transfer from one to another.

In the regions $\xi \neq 0$ we can write a cylindrically symmetric system in terms of another set of variables, which are useful in analyzing trajectories

$$\begin{aligned}
r &= (x_1^2 + x_2^2)^{1/2}, \qquad u = (v_1^2 + v_2^2)^{1/2}, \\
\phi &= \cos^{-1}((x_1 v_1 + x_2 v_2)/ru), \ (x_1 v_2 - x_2 v_1) = \xi > 0 \\
&= -\cos^{-1}((x_1 v_1 + x_2 v_2)/ru), \ (x_1 v_2 - {}_2 v_1) = \xi < 0 \\
z &= x_3, \qquad v_z = v_3.
\end{aligned} \tag{3.5}$$

The angle ϕ is between vectors (x_1, x_2) and (v_1, v_2) and takes on values from $-\pi$ to π. In terms of (3.5) the transport equation (3.4) is

$$\begin{aligned}
&\frac{\partial F}{\partial t} + u \cos \phi \frac{\partial F}{\partial r} + \left[-\frac{\partial \psi / \partial r}{u} - u/r \right] \sin \phi \frac{\partial F}{\partial \phi} \\
&\qquad + \frac{\partial \psi}{\partial r} \cos \phi \frac{\partial F}{\partial u} + v_z \frac{\partial F}{\partial z} + \frac{\partial \psi}{\partial z} \frac{\partial F}{\partial v_z} = 0.
\end{aligned} \tag{3.6a}$$

Since $\xi(t) = r(t)\, u(t) \sin \phi(t) = a$ along a trajectory of (3.6a), then if $\xi(0) > 0$ a trajectory of (3.6a) is confined to the region $r > 0$, $u > 0$, $0 < \phi < \pi$ if $\xi(0) < 0$ to the region $r > 0$, $u > 0$, $-\pi < \phi < 0$.

We can solve a somewhat less general cylindrically symmetric problem in terms of variables (3.5). That is we exclude from the model particles traveling in the planes containing the $r = 0$ axis. We define $D \subseteq R_5$ as

$$D = \{(r, u, \phi, z, v_z)/r \geqslant 0, u \geqslant 0, -\pi \leqslant \phi \leqslant \pi\}$$

and identify

$$p = (r, u, \phi, z, v_z)$$

as a point in D. The coordinate transformation (3.5) is written as the vector valued function

$$p(x, v) \colon R_6 \to D.$$

Let D_0 be a set in D of the form

$$D_0 = \{p \in D/r \geqslant d, u \leqslant d^{-1}, d \leqslant |\phi| \leqslant \pi - d, |z|, |v_z| \leqslant d^{-1}\}$$

for a small number d. By definition it will be said that a set is bounded in D if it is of the form D_0. We assume initial data g of the form

$$g(x, v) = G(p(x, v)) \tag{3.7}$$

for the function $G(p)$ of class $C'(D)$ and having support in D_0. For such a G

the function g is of class $C_0'(R_6)$. The solution f to (1.1a, b) is a function of the form

$$f(x, v) = F(p(x, v), t),$$

where F satisfies (3.6a) coupled with

$$\frac{\partial^2 \psi}{\partial r^2} + 1/r \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = -4\pi^2 \int Fu\, du\, d\phi\, dv_z, \qquad F = G \quad \text{at} \quad t = 0. \tag{3.6b}$$

System (3.6a,b) with initial data G models a system of particles all of which have angular momentum bounded away from zero.

According to Theorem 2.1, we can prove classical solvability of system (1.1a,b) by producing an a priori bound on the support of C' solutions to the system. Proofs for the cylindrically symmetric problem are given in Section 5. The a priori bounds are obtained by analyzing the characteristic system

$$\dot{r} = u \cos \phi, \qquad \dot{u} = \frac{\partial \psi}{\partial r} \cos \phi$$

$$\dot{\phi} = \left[-\frac{\partial \psi / \partial r}{u} - u/r \right] \sin \phi, \qquad \dot{z} = v_z, \qquad \dot{v}_z = \frac{\partial \psi}{\partial z}. \tag{3.8}$$

Condition I is the statement that

$$r(t)\, u(t) \sin \phi(t) = a \tag{3.9}$$

is an integral of (3.8). We first solve the problem for initial data of the form (3.7) and get bounds on support for this case where angular momentum is bounded away from zero. Then the problem for data of the form (3.2) is solved by getting another set of estimates for solutions to (3.8) in which angular momentum in the system is not restricted.

4. Bounds for the Field with Cylindrical Symmetry

The proofs of global-in-time solvability of system (1.1a,b) with data of the form (3.2), (3.7) depend on estimates for the cylindrically symmetric potential function. Let $h(x) \geqslant 0$ be a function of the form

$$h(x) = \bar{h}(r, z), \qquad r = (x_1^2 + x_2^2)^{1/2}, \qquad z = x_3.$$

Let $\psi(r, z)$ be the solution to

$$\frac{\partial^2 \psi}{\partial r^2} + 1/r \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = -4\pi^2 \bar{h}(r, z)$$

such that $\psi(r, z) \to 0$ as $r, |z| \to \infty$. Then

$$\frac{\partial \psi}{\partial r}(r, z) = \int \frac{\bar{h}(r - \bar{r}\cos\theta)\, \bar{r}\, d\bar{r}\, d\theta\, d\bar{z}}{(r^2 + \bar{r}^2 - 2r\bar{r}\cos\theta + (z - \bar{z})^2)^{3/2}}$$

$$\frac{\partial \psi}{\partial z}(r, z) = \int \frac{\bar{h}(z - \bar{z})\, \bar{r}\, d\bar{r}\, d\theta\, d\bar{z}}{(r^2 + \bar{r}^2 - 2r\bar{r}\cos\theta + (z - \bar{z})^2)^{3/2}}.$$

We write

$$I(r, \bar{r}, z, \bar{z}) = \int_0^{2\pi} (\frac{(r - \bar{r}\cos\theta)}{(r^2 + \bar{r}^2 + (z - \bar{z})^2 - 2r\bar{r}\cos\theta)^{3/2}}$$

$$= \frac{r}{((r - \bar{r})^2 + (z - \bar{z})^2)^{1/2}} \frac{i(r, \bar{r}, z, \bar{z})}{((r + \bar{r})^2 + (z - \bar{z})^2)},$$

$$i(r, \bar{r}, z, \bar{z}) = \int_0^{2\pi} \frac{(1 - \bar{r}/r\cos\theta)}{\left[\begin{array}{l}(1 + [2r\bar{r}/((r - \bar{r})^2 + (z - \bar{z})^2)](1 - \cos\theta))^{1/2} \\ \qquad \times (1 - [2r\bar{r}/((r + \bar{r})^2 + (z - \bar{z})^2)](1 + \cos\theta))\end{array}\right]},$$

$$J(r, \bar{r}, z, \bar{z}) = \int_0^{2\pi} \frac{(z - \bar{z})}{(r^2 + \bar{r}^2 + (z - \bar{z})^2 - 2r\bar{r}\cos\theta)^{3/2}}$$

$$= \frac{1}{\sqrt{(r - \bar{r})^2 + (z - \bar{z})^2}} \frac{j(r, \bar{r}, z, \bar{z})}{\sqrt{(r + \bar{r})^2 + (z - \bar{z})^2}},$$

$$j(r, \bar{r}, z, \bar{z}) = \frac{4(z - \bar{z})\, E(k)}{\sqrt{(r - \bar{r})^2 + (z - \bar{z})^2}},$$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2\sin^2\theta\, d\theta}, \qquad k = \sqrt{\frac{4r\bar{r}}{(r + \bar{r})^2 + (z - \bar{z})^2}},$$

$$\frac{\partial \psi}{\partial r}(r, z) = \int \bar{h} I(r, \bar{r}, z, \bar{z})\, \bar{r}\, d\bar{r}\, d\bar{z},$$

$$= \int \frac{\bar{h} r(i(r, \bar{r}, z, \bar{z}))\, \bar{r}\, d\bar{r}\, d\bar{z}}{((r - \bar{r})^2 + (z - \bar{z})^2)^{1/2}\, ((r + \bar{r})^2 + (z - \bar{z})^2)},$$

$$\frac{\partial \psi}{\partial z}(r, z) = \int \bar{h} J(r, \bar{r}, z, \bar{z})\, \bar{r}\, d\bar{r}\, d\bar{z}$$

$$= \int \frac{\bar{h} j(r, \bar{r}, z, \bar{z})\, \bar{r}\, d\bar{r}\, d\bar{z}}{((r - \bar{r})^2 + (z - \bar{z})^2)^{1/2}\, ((r + \bar{r})^2 + (z - \bar{z})^2)^{1/2}},$$

We show that the functions $i(r, \bar{r}, z, \bar{z})$, $j(r, \bar{r}, z, \bar{z})$ are bounded. This result is trivial for $j(r, \bar{r}, z, \bar{z})$. Clearly,

$$|j(r, \bar{r}, z, \bar{z})| \leqslant 4(\pi/2) = 2\pi.$$

Consider then the function $i(r, \bar{r}, z, \bar{z})$. Assume $z = 0$ and let $\bar{r} = \alpha r$, $\alpha \geqslant 0$. We bound the function $i(r, \alpha r, \bar{z}) = i(r, \alpha r, 0, \bar{z})$,

$$i(r, \alpha r, \bar{z}) = \int_0^{2\pi} \frac{(1 - \alpha \cos \theta)}{\left[\begin{array}{c}(1 + [2r^2\alpha/(r^2(1-\alpha)^2 + \bar{z}^2)](1 - \cos\theta))^{1/2} \\ \times (1 - [2r^2\alpha/(r^2(1+\alpha)^2 + \bar{z}^2)](1 + \cos\theta))\end{array}\right]}.$$

For relatively small values of α. The following estimate suffices:

Let $|\bar{z}| = \beta r$

$$\begin{aligned}
&|i(r, \alpha r, \bar{z})| \\
&\quad\leqslant \int \frac{|((1-\alpha)^2 + \beta^2)^{1/2}\, ((1+\alpha)^2 + \beta^2)(1 - \alpha\cos\theta)|}{(1 + \alpha^2 + \beta^2 - 2\alpha\cos\theta)^{3/2}} \\
&\quad\leqslant \int \frac{(|1-\alpha| + \beta)((1+\alpha)^2 + \beta^2)\,|1 - \alpha\cos\theta|}{(1 + \alpha^2 + \beta^2 - 2\alpha\cos\theta)^{3/2}}\, d\theta \\
&\quad\leqslant \int \frac{(|1-\alpha|\,(1+\alpha)^2 + \beta(1+\alpha)^2 + \beta^2\,|1-\alpha| + \beta^3)\,|1 - \alpha\cos\theta|}{(1 + \alpha^2 + \beta^2 - 2\alpha\cos\theta)^{3/2}}\, d\theta.
\end{aligned}$$

Now

$$\int \frac{\beta^3\,|1 - \alpha\cos\theta|}{(1 + \alpha^2 + \beta^2 - 2\alpha\cos\theta)^{3/2}}\, d\theta \leqslant 2\pi(1 + \alpha)$$

and

$$\begin{aligned}
&\int \frac{(|1-\alpha|\,(1+\alpha)^2 + \beta(1+\alpha)^2 + \beta^2\,|1-\alpha|)\,|1 - \alpha\cos\theta|}{(1 + \alpha^2 + \beta^2 - 2\alpha\cos\theta)^{3/2}}\, d\theta \\
&\quad\leqslant \int \frac{(|1-\alpha|\,(1+\alpha)^2 + \beta(1+\alpha)^2 + \beta^2\,|1-\alpha|)}{(1 + \alpha^2 + \beta^2 - 2\alpha\cos\theta)}\, d\theta \\
&\quad= \frac{(|1-\alpha|\,(1+\alpha)^2 + \beta(1+\alpha)^2 + \beta^2\,|1-\alpha|)}{\sqrt{(1-\alpha^2)^2 + 2(1+\alpha^2)\beta^2 + \beta^4}}\, 2\pi \\
&\quad\leqslant \left(\frac{|1-\alpha|\,(1+\alpha)^2}{|1-\alpha^2|} + \frac{\beta(1+\alpha)^2}{\beta\sqrt{2(1+\alpha^2)}} + \frac{\beta^2\,|1-\alpha|}{\beta^2}\right) 2\pi \\
&\quad\leqslant 6\pi(1 + \alpha).
\end{aligned}$$

Thus

$$|i(r, \alpha r, \bar{z})| \leqslant 8\pi(1+\alpha). \tag{4.1}$$

Assume now that $\alpha \geqslant 10$.

Let

$$\rho_1 = 2r^2\alpha(1-\cos\theta)/(r^2(1-\alpha)^2+\bar{z}^2)$$
$$\rho_2 = 2r^2\alpha(1+\cos\theta)/(r^2(1+\alpha)^2+\bar{z}^2).$$

Then

$$|\rho_1| \leqslant 4\alpha/(1-\alpha)^2 < 1/2, \qquad |\rho_2| \leqslant 4\alpha/(1+\alpha)^2 < 1/2$$

$$i(r, \alpha r, \bar{z}) = \int_0^{2\pi} \frac{(1-\alpha\cos\theta)}{(1+\rho_1)^{1/2}(1-\rho_2)}\, d\theta.$$

We write

$$(1+\rho_1)^{-1/2} = 1 + a_1\rho_1 + a_2\rho_1^2 + \cdots = 1 + S_1, \qquad |a_i| < 1,$$
$$(1-\rho_2)^{-1} = 1 + \rho_2 + \rho_2^2 + \cdots = 1 + S_2,$$
$$i(r, \alpha r, \bar{z}) = \int_0^{2\pi} \frac{d\theta}{(1+\rho_1)^{1/2}(1-\rho_2)} - \int_0^{2\pi} \alpha\cos\theta(1+S_1)(1+S_2)\, d\theta,$$
$$= \int_0^{2\pi} \frac{d\theta}{(1+\rho_1)^{1/2}(1-\rho_2)} - \int_0^{2\pi} \alpha\cos\theta(S_1+S_2+S_1S_2)\, d\theta.$$

Thus

$$|i(r, \alpha r, \bar{z})| \leqslant \int_0^{2\pi} \frac{d\theta}{(1+\rho_1)^{1/2}(1-\rho_2)}$$
$$+ \int_0^{2\pi} \alpha\,|\cos\theta|\,(|S_1| + |S_2| + |S_1S_2|)\, d\theta$$
$$\leqslant 4\pi + \int_0^{2\pi} \alpha\,|\cos\theta|\,(|S_1| + |S_2| + |S_1S_2|)\, d\theta.$$

However,

$$|S_1| \leqslant \rho_1[|a_1| + |a_2|\rho_1 + |a_3|\rho_1^2 + \cdots]$$
$$\leqslant \rho_1[1 + \rho_1 + \rho_1^2 + \cdots]$$
$$\leqslant \rho_1[1 + 1/2 + (1/2)^2 + \cdots] \leqslant 2\rho_1 \leqslant 8\alpha/(1-\alpha)^2$$
$$|S_2| \leqslant \rho_2[1 + (1/2) + (1/2)^2 + \cdots] \leqslant 2\rho_2 \leqslant 8\alpha/(1+\alpha)^2$$

and

$$|S_1 S_2| \leqslant (8\alpha)^2/[(1-\alpha)^2\,(1+\alpha)^2].$$

Thus

$$\begin{aligned}|S_1|+|S_2|+|S_1 S_2| &\leqslant 8\alpha/(1-\alpha)^2 + 8\alpha/(1+\alpha)^2 + 8\alpha^2/[(1-\alpha)^2\,(1+\alpha)^2]\\ &\leqslant 24\alpha/(1-\alpha)^2.\end{aligned}$$

We therefore have that

$$\begin{aligned}|i(r, \alpha r, \bar{z})| &\leqslant 4\pi + \frac{24\alpha^2}{(1-\alpha)^2}\int_0^{2\pi} |\cos\theta|\, d\theta\\ &\leqslant 4\pi + 96\alpha^2/(1-\alpha)^2, \qquad \alpha \geqslant 10. \qquad (4.2)\end{aligned}$$

Setting $\alpha = 10$ in (4.1), (4.2) we compute that

$$\begin{aligned}|i(r, \alpha r, 0, \bar{z}| &= |i(r, \alpha r, \bar{z})|\\ &\leqslant \max(8\pi(11), 4\pi + 96(100)/81) \leqslant 88\pi.\end{aligned}$$

Clearly, the same estimate applies for $z \neq 0$. We thus have a constant $D \leqslant 88\pi$ such that

$$|i(r, \bar{r}, z, \bar{z})| \leqslant D, \qquad |j(r, \bar{r}, z, \bar{z})| \leqslant D.$$

It follows that

$$\begin{aligned}\left|\frac{\partial\psi}{\partial r}\right| &\leqslant D\int \frac{|\bar{h}|\, r\bar{r}\, d\bar{r}\, d\bar{z}}{((r-\bar{r})^2+(z-\bar{z})^2)^{1/2}\,((r+\bar{r})^2+(z-\bar{z})^2)}\\ \left|\frac{\partial\psi}{\partial z}\right| &\leqslant D\int \frac{|\bar{h}|\, \bar{r}\, d\bar{r}\, d\bar{z}}{((r-\bar{r})^2+(z-\bar{z})^2)^{1/2}\,((r+\bar{r})^2+(z-\bar{z})^2)^{1/2}}. \qquad (4.3)\end{aligned}$$

Using expressions (4.3) estimates are now obtained for the derivatives of the potential function. Assume $\bar{h}$ has bounds

$$\sup_{R_3} |\bar{h}| \leqslant A, \qquad (4.4\text{i})$$

$$\int \bar{h}\bar{r}\, d\bar{r}\, d\bar{z} = M, \qquad (4.4\text{ii})$$

$$\left(\int \bar{h}^s \bar{r}\, d\bar{r}\, d\bar{z}\right)^{1/s} \leqslant K, \qquad 1 < s \leqslant 5/3, \qquad (4.4\text{iii})$$

$$\int \bar{r}^2\, |\bar{h}|\, \bar{r}\, d\bar{r}\, d\bar{z} \leqslant C. \qquad (4.4\text{iv})$$

We consider first the radial component of the field. Let

$$\rho = ((r-\bar{r})^2 + (z-\bar{z})^2)^{1/2},$$

$$1/D\left|\frac{\partial\psi}{\partial r}\right| \leqslant r\int_{\rho\leqslant\varepsilon}\frac{\bar{h}\bar{r}\,d\bar{r}\,d\bar{z}}{((r-\bar{r})^2)+(z-\bar{z})^2)^{1/2}\,((r+\bar{r})^2+(z-\bar{z})^2)} + r\int_{\rho\geqslant\varepsilon}\frac{\bar{h}\bar{r}\,d\bar{r}\,d\bar{z}}{((r-\bar{r})^2+(z-\bar{z})^2)^{1/2}\,((r+\bar{r})^2+(z-\bar{z})^2)}$$

$$r\int_{\rho\leqslant\varepsilon}\frac{\bar{h}\bar{r}\,d\bar{r}\,d\bar{z}}{((r-\bar{r})^2+(z-\bar{z})^2)^2)^{1/2}\,((r+\bar{r})^2+(z-\bar{z})^2)} \leqslant \int_{\rho\leqslant\varepsilon}\frac{|\bar{h}|\,d\bar{r}\,d\bar{z}}{\rho} \leqslant A\int_{\rho\leqslant\varepsilon}\frac{\rho\,d\rho\,d\beta}{\rho} \leqslant 2\pi A\varepsilon.$$

Thus

$$1/D\left|\frac{\partial\psi}{\partial r}\right| \leqslant 2\pi A\varepsilon + r\int_{\rho\geqslant\varepsilon}\frac{\bar{h}\bar{r}\,d\bar{r}\,d\bar{z}}{\rho((r+\bar{r})^2+(z-\bar{z})^2)}.$$

Depending on how the integral for $\rho\geqslant\varepsilon$ is estimated one obtains several different bounds for $|\partial\psi/\partial r|$.

A bound for r near 0:

$$\begin{aligned} r\int_{\rho\geqslant\varepsilon}&\frac{\bar{h}\bar{r}\,d\bar{r}\,d\bar{z}}{\rho((r+\bar{r})^2+(z-\bar{z})^2)} \\ &\leqslant r\left(\int \bar{h}^{5/3}\bar{r}\,d\bar{r}\,d\bar{z}\right)^{3/5}\left(\int_{\rho\geqslant\varepsilon}\frac{\bar{r}\,d\bar{r}\,d\bar{z}}{\rho^{5/2}((r+\bar{r})^2+(z-\bar{z})^2)^{5/2}}\right)^{2/5} \\ &\leqslant rK\left(\int\frac{d\bar{r}\,d\bar{z}}{\rho^{5/2}(\rho^2+4r\bar{r})^2}\right)^{2/5} \\ &\leqslant rK\left(\int_{\rho\geqslant\varepsilon}\frac{\rho\,d\rho\,d\beta}{\rho^{13/2}}\right)^{2/5} \leqslant (2/9)^{2/5}\,\frac{(2\pi)^{2/5}\,Kr}{\varepsilon^{9/5}}. \end{aligned}$$

Hence,

$$1/D\,|\partial\psi/\partial r| \leqslant 2\pi A\varepsilon + (2/9)^{2/5}\,((2\pi)^{2/5}\,Kr/\varepsilon^{9/5}).$$

Setting

$$\varepsilon = ((2/9)^{2/5}/(2\pi)^{3/5})(Kr/A)^{5/14}.$$

We get the estimate

$$|\partial\psi/\partial r| \leqslant D\,2(4\pi/9)^{2/5}\,K^{5/14}A^{9/14}r^{5/14} \leqslant \alpha K^{5/14}A^{9/14}r^{5/14}. \tag{4.5}$$

(Here and in following estimates α is some sufficiently large constant.) An estimate for intermediate values of r:

$$
\begin{aligned}
r \int_{\rho \geqslant \varepsilon} & \frac{\bar{h}\bar{r}\, d\bar{r}\, d\bar{z}}{\rho((r+\bar{r})^2+(z-\bar{z})^2)} \\
&\leqslant r \left(\int \bar{h}^{5/3}\right)^{3/5} \left(\int_{\rho \geqslant \varepsilon} \frac{\bar{r}\, d\bar{r}\, d\bar{z}}{\rho^{5/2}((r+\bar{r})^2+(z-\bar{z})^2)^{5/2}}\right)^{2/5} \\
&\leqslant rK \left(\int \frac{d\bar{r}\, d\bar{z}}{\rho^{5/2}((r+\bar{r})^2+(z-\bar{z})^2)^2}\right)^{2/5} \\
&\leqslant rK \frac{1}{r^{8/5}} \left(\int_{\rho \geqslant \varepsilon} \frac{\rho\, d\rho\, d\beta}{\rho^{5/2}}\right)^{2/5} \\
&\leqslant 2^{2/5}((2\pi)^{2/5}K/(r^{3/5}\varepsilon^{1/5})).
\end{aligned}
$$

Thus

$$1/D\,|\partial\psi/\partial r| \leqslant 2\pi A\varepsilon + 2^{2/5}(2\pi)^{2/5}K/(r^{3/5}\varepsilon^{1/5})$$

for $\varepsilon = (2^{1/3}/(2\pi)^{1/2})(K^{5/6}/(A^{5/6}r^{1/2}))$,

$$|\partial\psi/\partial r| \leqslant D\, 2(2)^{1/3}((2\pi)^{1/2}K^{5/6}A^{1/6}/r^{1/2}) \leqslant \alpha K^{5/6}A^{1/6}/r^{1/2}. \tag{4.6}$$

An estimate for large values of r:

$$
\begin{aligned}
r \int_{\rho \geqslant \varepsilon} \frac{\bar{h}\bar{r}\, d\bar{r}\, d\bar{z}}{\rho((r+\bar{r})^2+(z-\bar{z})^2)} &\leqslant 1/r \int_{\varepsilon \leqslant \rho \leqslant 1} \frac{\bar{h}\bar{r}\, d\bar{r}\, d\bar{z}}{\rho} + 1/r \int_{\rho \geqslant 1} \frac{\bar{h}\bar{r}\, d\bar{r}\, d\bar{z}}{\rho} \\
&\leqslant 1/r \int_{\varepsilon \leqslant \rho \leqslant 1} \frac{\bar{h}\bar{r}\, d\bar{r}\, d\bar{z}}{\rho} + M/r \\
\int_{\varepsilon \leqslant \rho \leqslant 1} \frac{\bar{h}\bar{r}\, d\bar{r}\, d\bar{z}}{\rho} &\leqslant \left(\int |\bar{h}|^2\, \bar{r}^2\, d\bar{r}\, d\bar{z}\right)^{1/2} \left(\int_{\varepsilon \leqslant \rho \leqslant 1} \frac{\rho\, d\rho\, d\beta}{\rho^2}\right)^{1/2} \\
&\leqslant \left(\int \bar{h}^2\bar{r}^2\, d\bar{r}\, d\bar{z}\right)^{1/2} (2\pi)^{1/2}\, (\ln(1/\varepsilon))^{1/2}.
\end{aligned}
$$

However,

$$
\begin{aligned}
\left(\int \bar{h}^2\bar{r}^2\, d\bar{r}\, d\bar{z}\right)^{1/2} &= \left(\int \bar{h}^{3/2}\bar{r}^{1/2}\bar{h}^{1/2}\bar{r}^{3/2}\, d\bar{r}\, d\bar{z}\right)^{1/2} \\
&\leqslant \left(\int \bar{h}^3\bar{r}\, d\bar{r}\, d\bar{z}\right)^{1/4} \left(\int \bar{h}\bar{r}^3\, d\bar{r}\, d\bar{z}\right)^{1/4} \\
&\leqslant C^{1/4} \left(\int (\sup \bar{h})^{4/3}\, \bar{h}^{5/3}\bar{r}\, d\bar{r}\, d\bar{z}\right)^{1/4} = C^{1/4}K^{5/12}A^{1/3}.
\end{aligned}
$$

Thus,

$$\begin{aligned}1/D\,|\partial\psi/\partial r| \leqslant 2\pi A\varepsilon + (2\pi)^{1/2}\,(C^{1/4}K^{5/12}A^{1/3}/r)\\ \times (\ln(1/\varepsilon))^{1/2} + M/r.\end{aligned}$$

Letting $\varepsilon = 1/(A^{2/3}r)$ we have

$$\begin{aligned}1/D\,|\partial\psi/\partial r| &\leqslant (2\pi)\,A^{1/3}/r + (2\pi)^{1/2}\,(C^{1/4}K^{5/12}A^{1/3}/r)\\ &\qquad \times (\ln(A^{2/3}r))^{1/2} + M/r\\ |\partial\psi/\partial r| &\leqslant D(6\pi)\,(C^{1/4}K^{5/12}A^{1/3}/r)\,\sqrt{\ln(A^{2/3}r)}\\ &\leqslant \alpha(C^{1/4}K^{5/12}A^{1/3}/r)\,\sqrt{\ln(A^{2/3}r)}\ (\text{assuming } r \geqslant 1/A^{2/3}). \qquad (4.7)\end{aligned}$$

Estimates (4.5), (4.6) are a natural reduction to cylindrical symmetry of estimate (2.4). They are computed from bounds (4.4i), (4.4iii) as is (2.4). As functions of r they intersect at $r = (K/A)^{5/9}$ at which point they have the value $\alpha K^{5/9}A^{4/9} = BA^{4/9}$, the form of estimate (2.4). The estimate (4.7) is dfferent. In addition to the bounds (4.4i), (4.4iii) we also make use of the bounds (4.4ii), (4.4iv) in computing this estimate. We note from Lemma (2.5) that the constant C depends on time.

We obtain the estimates for $\partial\psi/\partial z$.

$$\begin{aligned}1/D\left|\frac{\partial\psi}{\partial z}\right| &\leqslant \int \frac{\bar h\bar r\,d\bar r\,d\bar z}{((r-\bar r)^2+(z-\bar z)^2)^{1/2}\,((r+\bar r)^2+(z-\bar z)^2)^{1/2}}\\ &= \int_{\rho<\varepsilon} \frac{\bar h\bar r\,d\bar r\,d\bar z}{((r-\bar r)^2+(z-\bar z)^2)^{1/2}\,((r+\bar r)^2+(z-\bar z)^2)^{1/2}}\\ &\quad + \int_{\rho\geqslant\varepsilon} \frac{\bar h\bar r\,d\bar r\,d\bar z}{((r-\bar r)^2+(z-\bar z)^2)^{1/2}\,((r+\bar r)^2+(z-\bar z)^2)^{1/2}}.\end{aligned}$$

As before

$$\int_{\rho<\varepsilon} \frac{\bar h r\bar r\,d\bar r\,d\bar z}{\rho((r+\bar r)^2+(z-\bar z)^2)^{1/2}} \leqslant 2\pi A\varepsilon$$

$$1/D\left|\frac{\partial\psi}{\partial z}\right| \leqslant 2\pi A\varepsilon + \int_{\rho\geqslant\varepsilon} \frac{\bar h\bar r\,d\bar r\,d\bar z}{\rho((r+\bar r)^2+(z-\bar z)^2)^{1/2}}.$$

For r near 0:

$$\int_{\rho\geqslant\varepsilon} \frac{\bar h\bar r\,d\bar r\,d\bar z}{\rho((r+\bar r)^2+(z-\bar z)^2)^{1/2}}$$

$$\leqslant \left(\int h^{5/3}\bar{r}\,d\bar{r}\,d\bar{z}\right)^{3/5} \left(\int \frac{\bar{r}\,d\bar{r}\,d\bar{z}}{\rho^{5/2}((r+\bar{r})^2+(z-\bar{z})^2)^{5/2}}\right)^{2/5}$$
$$\leqslant K\left(\int \frac{\rho\,d\rho\,d\beta}{\rho^4}\right)^{2/5} = K\frac{(2\pi)^{2/5}}{2^{2/5}}\,1/\varepsilon^{4/5}.$$

Thus

$$1/D\,|\partial\psi/\partial z| \leqslant 2\pi A\varepsilon + (K(2\pi)^{2/5}/2^{2/5})\,1/\varepsilon^{4/5}.$$

For $\varepsilon = ((K)^{5/9}/(2\pi)^{1/3})(1/2^{2/9})\,1/A^{5/9}$,

$$|\partial\psi/\partial z| \leqslant D(2/2^{2/9})(2\pi)^{2/3}\,K^{5/9}A^{4/9} \leqslant \alpha K^{5/9}A^{4/9} \tag{4.8}$$

(estimate (2.4)). For intermediate values of r:

$$\int_{\rho\geqslant\varepsilon} \frac{\bar{h}\bar{r}\,d\bar{r}\,d\bar{z}}{\rho((r+\bar{r})^2+(z-\bar{z})^2)^{1/2}}$$
$$\leqslant \left(\int \bar{h}^{5/3}\bar{r}\,d\bar{r}\,d\bar{z}\right)^{3/5} \left(\int \frac{\bar{r}\,d\bar{r}\,d\bar{z}}{\rho^{5/2}((r+\bar{r})^2+(z-\bar{z})^2)^{1/2})^{5/2}}\right)^{2,5}$$
$$\leqslant K(2)^{2/5}\frac{1}{r^{3/5}}\left(\int_{\rho\geqslant\varepsilon} \frac{\rho\,d\rho\,d\beta}{\rho^{5/2}}\right)^{2/5}$$
$$= (K(2\pi)^{2/5}\,2^{2/5}/(r^{3/5}\varepsilon^{1/5}))$$
$$1/D\,|\partial\psi/\partial z| \leqslant 2\pi A\varepsilon + (K(2\pi)^{2/5}\,2^{2/5}/(r^{3/5}\varepsilon^{1/5}).)$$

As in (4.6), setting $\varepsilon = 2^{1/3}K^{5/6}/((2\pi)^{1/2}\,A^{5/6}r^{1/2})$ yields the estimate

$$|\partial\psi/\partial z| \leqslant (D2(2)^{1/3}\,(2\pi)^{1/2}\,K^{5/6}A^{1/6})/r^{1/2} \leqslant \alpha K^{5/6}A^{1/6}/r^{1/2}. \tag{4.9}$$

For large values of r:

$$1/D\left|\frac{\partial\psi}{\partial z}\right| \leqslant 2\pi A\varepsilon + \int_{\rho\geqslant\varepsilon} \frac{\bar{h}\bar{r}\,d\bar{r}\,d\bar{z}}{\rho((r+\bar{r})^2+(z-\bar{z})^2)^{1/2}}$$
$$\leqslant 2\pi A\varepsilon + 1/r\int_{\varepsilon\leqslant\rho\leqslant 1} \frac{\bar{h}\bar{r}\,d\bar{r}\,d\bar{z}}{\rho} + M/r.$$

At this point the computation is the same as for $\partial\psi/\partial r$. Thus

$$|\partial\psi/\partial z| \leqslant D6\pi C^{1/4}K^{5/12}A^{1/3}\sqrt{\ln(A^{2/3}r)}/r$$
$$\leqslant \alpha C^{1/4}K^{5/12}A^{1/3}\sqrt{\ln(A^{2/3}r)}/r. \tag{4.10}$$

As a summary of the results of this section estimates (4.5)–(4.10) are pieced together into continuous functions of r which bound the r, z

components of the electric (gravitational) field. Let $\eta = (K/A)^{5/9}$ and l, the solution to

$$x = (C^{1/2}A^{1/3}/K^{5/6})\ln(A^{2/3}x).$$

The functions are

$$\begin{aligned}
w_r(r) &= \alpha K^{5/14}A^{9/14}r^{5/14}, && r \leqslant \eta,\\
&= (\alpha K^{5/6}A^{1/6})/r^{1/2}, && \eta \leqslant r \leqslant l,\\
&= \alpha(C^{1/4}K^{5/12}A^{1/3}\sqrt{\ln(A^{2/3}r)})/r && r \geqslant l.\\
w_z(r) &= \alpha K^{5/9}A^{4/9}, && r \leqslant \eta,\\
&= (\alpha K^{5/6}A^{1/6})/r^{1/2}, && \eta \leqslant r \leqslant l,\\
&= \alpha(C^{1/4}K^{5/12}A^{1/3}\sqrt{\ln(A^{2/3}r)})/r, && r \geqslant l.
\end{aligned}$$

(Note that $\alpha K^{5/14}A^{9/14}\eta^{5/14} = (\alpha K^{5/6}A^{1/6})/\eta^{1/2} = \alpha K^{5/9}A^{4/9}$.) The bound is

$$|\partial\psi/\partial r| \leqslant w_r(r), \qquad |\partial\psi/\partial z| \leqslant w_z(r).$$

5. Existence and Uniqueness of Cylindrical Solutions

In this section the statements and proofs of theorems on global-in-time existence and uniqueness of cylindrically symmetric solution to (1.1a,b) are given. Two theorems are proved. The first, Theorem 5.1, is for initial data of the form (3.7). This is a restricted set of data for which the system (1.1a,b) can be written in terms of the variables (3.5). A priori estimates on support are obtained by integrating the characteristic system (3.8). We compute estimates in this case which include a lower bound on the angular momentum in the system and are therefore only applicable for data of the form (3.7). The second theorem is for more general cylindrical data of the form (3.2). The proof is a slight modification and extension of that for Theorem 5.1. A second set of estimates for (3.8) are obtained which do not depend on angular momentum.

In order to state Theorem 5.1 the set D_0 in Section 3 is defined more precisely in terms of constants R_0, Q_0, δ, P_0 as

$$D_0 = \left\{ P \in D \middle/ \begin{array}{c} R_0^{-1} \leqslant r \leqslant R_0 \\ Q_0^{-1/2} \leqslant u \leqslant Q_0^{1/2} \\ \delta \leqslant |\phi| \leqslant \pi - \delta \\ |z| \leqslant R_0, |v_z| \leqslant P_0 \end{array} \right\} \tag{5.1}$$

The set E_0 is then defined as the set in R_6 such that

$$p(E_0) = D_0. \tag{5.2}$$

With the definition of D_0 and E_0 given by (5.1), (5.2), the definition of D given in Section 3 an existence and uniqueness result for cylindrically symmetric solutions is

THEOREM 5.1. *Let $G(p)$ be a function of class $C'(D)$ having support in D_0. Assume the initial data g to* (1.1a, b) *is of the form*

$$g(x, v) = G(p(x, v)).$$

Then g is symmetric initial data of class $C_0'(R_6)$ which has support in E_0. System (1.1a, b) *with initial data g has a unique global-in-time classical solution. The solution is of the form*

$$f(x, v, t) = F(p(x, v), t),$$

where $F(p, t)$ has bounded support in D for each t and is a classical solution to (3.6a, b) *with initial data G.*

Proof. If f is a C' solution to (1.1a, b) with bounded support, then f can be constructed iteratively as in the proof of Theorem 2.1. It is easy to see that f satisfies conditions I, II (Section 3). In terms of the cylindrical variables of Section 3 condition I is the statement that $\xi(t) = r(t)\, u(t) \sin\phi(t) = a$ (const) along trajectories. If f has compact support, then $r(t)$, $u(t)$ are bounded from above. Also, by virtue of condition I, $\xi(t)$ is bounded away from zero for points in the support of f. Hence $r(t)$, $u(t)$ are bounded away from zero, $\phi(t)$ bounded away from 0, π. The solution f is necessarily of the form

$$f(x, v, t) = F(p(x, v), t),$$

where $F(p, t)$ is a solution to (3.6a, b) having bounded support in D (a bounded set in D is of the form D_0).

To demonstrate the existence of an a priori bound for the support of the solution to (1.1a, b) we analyze the system (3.6) and in particular the characteristic system (3.8). What we show is that for arbitrary $T > 0$ the solutions to (3.8) originating at points $p \in \operatorname{supp} G$ are uniformly bounded for $t \in [0, T]$.

For F a solution to (3.6a, b) we define the following sets:

$$A(t) = \operatorname{supp} F(\cdot, t) \qquad \text{the support of } F \text{ in } D \text{ at time } t$$

$$A(T) = \bigcup_t A(t) \qquad \text{for} \quad t \in [0, T].$$

For $p \in D$ let

$$E_1(p) = v_z, \qquad E_2(p) = u, \qquad E_3(p) = r$$

be the projections onto the respective axis. The constants P, Q, R are defined as

$$P = \sup_{A(T)} (|E_1|), \qquad Q = \sup_{A(T)} (E_2^2), \qquad R = \sup_{A(T)} (E_3). \tag{5.3}$$

We assume in this section that the initial data $g(x, v) = G(p(x, v))$ to (1.1a, b) is non-negative and bounded according to

$$\sup |g| = M_0, \qquad \|g\|_1 = 2\pi M, \qquad \|g\|_p = (2\pi)^{1/p} M(p).$$

The function $G(p)$ therefore has the bounds

$$\sup G(p) = M_0, \tag{5.4i}$$

$$\int Gu\,du\,d\phi\,dv_z r\,dr\,dz = M, \tag{5.4ii}$$

$$\left(\int G^p u\,du\,d\phi\,dv_z r\,dr\,dz\right)^{1/p} = M(p). \tag{5.4iii}$$

The function $F(p)$ which is taken to be the solution to (3.6) has bounds

$$\sup F(p) = M_0,$$

$$\int Fu\,du\,d\phi\,dv_z r\,dr\,dz = M,$$

$$\left(\int F^p u\,du\,d\phi\,dv_z r\,dr\,dz\right)^{1/p} = M(p).$$

Let

$$\bar{h} = \int Fu\,du\,d\phi\,dv_z.$$

From Lemmas (2.4), (2.5) it can be assumed that $\bar{h}$ has bounds

$$\left(\int \bar{h}^{5/3} r\,dr\,dz\right)^{3/5} \leqslant K, \tag{5.5i}$$

$$\int \bar{h} r^3\,dr\,dz \leqslant C(T) = C(1+T)^2. \tag{5.5ii}$$

From definition (5.3) we have that

$$\sup|\tilde{h}| \leqslant \pi M_0(PQ).$$

Let ψ be the solution to (3.6b). Various estimates on the derivative of ψ are computed in Section 4. We use only expressions (4.6), (4.9) for intermediate values of r as a bound for the field. As has been pointed out this bound (along with (4.5)) is a refinement due to symmetry of the bound (2.4). We are not making use of integral bound (5.5ii) in using only estimate (4.6), (4.9) for the field. A more precise bound on support can be computed using combinations of the estimates (4.5)–(4.10). Let

$$\beta = \alpha K^{5/6}(\pi M_0)^{1/6}, \qquad w(r) = \beta(PQ)^{1/6}/r^{1/2}, \qquad W(r) = 2\beta(PQ)^{1/6}\, r^{1/2}.$$

A bound for the field is

$$|\partial\psi/\partial r|, \qquad |\partial\psi/\partial z| \leqslant w(r) = W'(r). \tag{5.6}$$

To get a bound on the support of the solution F to (3.6a, b) we consider a trajectory $p(t)$ (solution to (3.8)) for $t \in [0, T]$ and such that $p(0) \in \operatorname{supp} G \subset D_0$. For such a trajectory we first show that

$$u^2(t) \leqslant Q_0 + 9\beta(PQ)^{1/6}\, R^{1/2} \tag{5.7a}$$

$$|v_z(t)| \leqslant P_0 + \beta/c\, P^{1/6} Q^{5/12} T \qquad c = R_0^{-1} Q_0^{-1/2} \sin(\delta). \tag{5.7b}$$

The computation of (5.7a) is carried out in several steps.

(i) For two points $p(t_1)$, $p(t_2)$, t_1, $t_2 \leqslant T$,

$$|W(r(t_2)) - W(r(t_1))| \leqslant 2\beta(PQ)^{1/6}\, R^{1/2}.$$

(ii) At a local maximum of $r(t)$,

$$\dot r = u\cos\phi = 0 \Rightarrow \phi = \pm\pi/2,$$

$$\ddot r = u\cos(\pm\pi/2) - u\sin(\pm\pi/2)\,\dot\phi < 0 \Rightarrow \dot\phi > 0, \qquad \text{if} \quad \phi = \pi/2,$$

$$\Rightarrow \dot\phi < 0, \qquad \text{if} \quad \phi = -\pi/2.$$

Hence from (3.8), (5.6), $\partial\psi/\partial r < 0$ and

$$u^2(t) \leqslant -(\partial\psi/\partial r)\, r \leqslant w(r)\, r \leqslant \beta(PQ)^{1/6}\, R^{1/2}.$$

(iii) At a value of t such that $r(t)$ is a local minimum there exists a $t_0 < t$ such that either $t_0 = 0$ or $r(t_0)$ is a local maximum and $\dot r \leqslant 0$ on the interval $[t_0, t]$. From (3.8) we get,

$$u\dot u = (\partial\psi/\partial r)\, u\cos\phi = (\partial\psi/\partial r)\,\dot r.$$

Hence

$$\begin{aligned}
u^2(t) &= u^2(t_0) + 2\int_{t_0}^{t} \frac{\partial\psi}{\partial r}\,\dot r\,dt \leqslant u^2(t_0) - 2\int_{t_0}^{t}\left|\frac{\partial\psi}{\partial r}\right|\dot r\,dt\\
&\leqslant u^2(t_0) - 2\int_{t_0}^{t}\frac{d}{dt}\,W(r(\bar t))\,d\bar t\\
&= u^2(t_0) + 2[W(r(t_0)) - W(r(t_1))]\\
&\leqslant Q_0 + \beta(PQ)^{1/6}R^{1/2} + 4\beta(PQ)^{1/6}R^{1/2}\\
&\leqslant Q_0 + 5\beta(PQ)^{1/6}R^{1/2}.
\end{aligned}$$

(iv) At any point $t \leqslant T$ there exists a $t_0 \leqslant t$ such that either $t_0 = 0$ or t_0 is a local extremum of $r(t)$ and $\dot r$ does not change sign on $[t_0, t]$. It follows that

$$\begin{aligned}
u^2(t) &\leqslant u^2(t_0) + 2\,|W(r(t)) - W(r(t_0))|\\
&= Q_0 + 5\beta(PQ)^{1/6}R^{1/2} + 4\beta(PQ)^{1/6}R^{1/2}\\
&\leqslant Q_0 + 9\beta(PQ)^{1/6}R^{1/2}.
\end{aligned}$$

This completes the computation of (5.7a).

The computation of (5.7b) is straightforward. Integrating (3.8) we get

$$v_z(t) = v_z(0) + \int_0^t \frac{\partial\psi}{\partial z}\,d\bar t$$

and therefore according to bound (4.9)

$$|v_z(t)| \leqslant P_0 + \beta\int_0^t \frac{(PQ)^{1/6}}{r(\bar t)^{1/2}}\,d\bar t.$$

But $u(t)\,r(t)\sin(\phi(t)) = \bar c$, where $\bar c = Q_0^{-1/2}R_0^{-1}\sin(\delta)$ so

$$\begin{aligned}
|v_z(t)| &\leqslant P_0 + \frac{\beta(PQ)^{1/6}}{c}\int_0^t (u(\bar t)\,|\sin(\phi(\bar t))|)^{1/2}\,d\bar t\\
&\leqslant P_0 + (\beta(PQ)^{1/6}/c)\,Q^{1/4}T\\
&\leqslant P_0 + (\beta P^{1/6}/c)\,Q^{5/12}T
\end{aligned}$$

which is the bound (5.7b).

We now carry out the remainder of the computation for the bound on the support of F. Integrating (3.8) and using definitions (5.3) we get

$$R \leqslant R_0 + Q^{1/2}T. \tag{5.8}$$

Since (5.7a, b) are satisfied along any trajectory $p(t)$ such that $p(0) \in \operatorname{supp} G$, then taking the supremum over all such trajectories we have that P, Q must satisfy inequalities

$$Q \leqslant Q_0 + 9\beta(PQ)^{1/6}(R_0 + Q^{1/2}T)^{1/2} \tag{5.9a}$$

$$P \leqslant P_0 + \beta/c\, P^{1/6}Q^{5/12}T. \tag{5.9b}$$

Solving (5.9b) for $P^{1/6}$ and substituting into (5.9a) one obtains an inequality for Q of the form

$$\begin{aligned} Q &\leqslant Q_0 + b_1 Q^{1/6} + b_2 Q^{1/4}T^{1/5} + b_3 Q^{5/12}T^{1/2} + b_4 Q^{1/2}T^{7/10} \\ &\leqslant Q_0 + bQ^{1/2}(1 + T^{7/10}) \\ b &= 4\max(b_i), \qquad i = 1,..., 4. \end{aligned} \tag{5.10}$$

The constants b_i depend on β, c, $R_0 P_0$. From this inequality a bound for Q is derived:

$$Q \leqslant 4(Q_0 + b^2)(1 + T^{7/5}).$$

Substituting this bound into (5.9b) gives an inequality which can be solved for P in terms of T. For a constant γ which depends on Q_0, P_0, b, β, c the following type bound for Q, P can be obtained:

$$Q \leqslant \gamma(1 + T^{7/5}), \qquad P \leqslant \gamma(1 + T^{19/10}). \tag{5.11}$$

The bound on R is obtained from (5.8). Given a trajectory $p(t)$ such that $p(0) \subseteq \operatorname{supp} G$, we thus have for $t \in [0, T]$

$$|v_z(t)| \leqslant P, \qquad |z(t)| \leqslant R_0 + PT, \qquad u(t) \leqslant Q^{1/2}, \qquad r(t) \leqslant R.$$

Thus for an arbitrary value of T the bound on support is obtained for (3.6a, b) with initial data G and likewise for (1.1a, b) with symmetric data $g = (p(x, v))$. From Theorem 2.1 it then follows that the global-in-time cylindrically symmetric solutions to (1.1a, b) exist and are unique.

The a priori estimates on support obtained in the proof of Theorem 5.1 do not suffice for the more general cylindrical data of the form (3.2) since a lower bound on the angular momentum in the system is used in obtaining the estimates. In the next theorem estimates are obtained which do not depend on the angular momentum. This allows us to extend our results to cylindrically symmetric data of the form (3.2), that is of the form where $s(x, v)$: $R_6 \to R_5$ is the transformation given by,

$$r = (x_1^2 + x_2^2)^{1/2}, \qquad \eta = (x_1 v_1 + x_2 v_2)$$

$$\xi = (x_1 v_2 - x_2 v_1), \qquad z = x_3, \qquad v_z = v_3.$$

THEOREM 5.2. *Let $g(x, v)$ be a function of class $C_0'(R_6)$ which is of the form* (3.2) *(cylindrically symmetric). The system* (1.1a,b) *with initial data g has a unique global-in-time classical solution.*

Proof. The proof is essentially that of Theorem 5.1 with a few modifications. Let f be a solution to (1.1a,b) with initial data g. If f is a function with compact support in R_6 for each t, then f can be constructed iteratively as in Theorem (2.1). Conditions I, II, Section 3, are satisfied and f is of the form (3.3). Let

$$A(t) = \operatorname{supp} f(\cdot, t) \qquad \text{the support of } f \text{ in } R_6 \text{ at time } t$$

$$A(T) = \bigcup_t A(t) \qquad \text{for } t \in [0, T]$$

$$r(x, v) = (x_1^2 + x_2^2)^{1/2}, \qquad u(x, v) = (v_1^2 + v_2^2)^{1/2}, \qquad v_z(x, v) = v_3.$$

The constants P, Q, R are defined as

$$P = \sup_{A(T)} |v_z(x, v)|, \qquad Q = \sup_{A(T)} (u(x, v))^2, \qquad R = \sup_{A(T)} (r(x, v)).$$

Now P_0, Q_0, R_0, the values of P, Q, R at $T = 0$, are bounds on the support of the initial data g. Let

$$\bar{h}(r, z) = h(x) = \int f\, dv.$$

We assume again that

$$\sup |f| \leqslant M_0, \qquad \left(\int \bar{h}^{5/3} r\, dr\, dz\right)^{3/5} \leqslant K.$$

Bounds on the field are obtained from Section 4. Let

$$|\partial\psi/\partial r| \leqslant w_r(r) = \beta(PQ)^{1/6}/r^{1/2}, \qquad |\partial\psi/\partial z| \leqslant w_z(r) = \beta(PQ)^{4/9},$$
$$\beta = \max(\alpha K^{5/9}(\pi M_0)^{4/9}, \alpha K^{5/6}(\pi M_0)^{1/6}).$$

Clearly, Q satisfies inequality (5.9a). We get this by integrating (3.8) as in the proof of Theorem 5.1 in the regions $\xi \neq 0$. The result is extended by continuity to include the planes for which $\xi = 0$ (or we could integrate a reduced set of characteristic equations in $\xi = 0$ planes as a part of the computation). Using the bound (4.8) and carrying out a similar computation as in Theorem 5.1 it can be shown that P satisfies

$$P \leqslant P_0 + \beta P^{4/9} Q^{4/9} T.$$

We get a bound on support in velocity space by solving the set of inequalities

$$Q \leqslant Q_0 + 9\beta(PQ)^{1/6}(R_0 + Q^{1/2}T)^{1/2}, \qquad P \leqslant P_0 + \beta P^{4/9}Q^{4/9}T.$$

A solution is

$$Q \leqslant \gamma(1 + T^{16/9}), \qquad P \leqslant \gamma(1 + T^{29/9}). \tag{5.12}$$

The constant γ here does not depend on a lower bound on angular momentum as it did in the proof of Theorem 5.1. From inequalities (5.12) we get an a priori bound on support of the solution f and the Theorem is proved.

In the proofs of Theorems 5.1 and 5.2 two different bounds (5.11), (5.12) on velocities in the system are obtained. Bound (5.11) has a lower power of T but the coefficient γ is proportional to $1/c$; c is the lower bound on angular momentum. Bound (5.12) has a higher power of T but γ is independent of c. The difference comes about in the integration of the z component of velocity. In Theorem 5.1 bound (4.6) is used in estimating this integral, in Theorem 5.2 bound (4.8).

REFERENCES

1. A. A. ARSENEV, Existence and uniqueness in the small for the classical solution of a system of Vlasov equations, *Akad. Nauk SSSR Dokl.* **28** (1974), 11–12.
2. J. BATT, Global symmetric solution of the inital value problem of stellar dynamics, *J. Differential Equations* **25** (1977), 342–364.
3. J. BATT, Recent developments in the mathematical investigation of the initial value problem of stellar dynamics and plasma, *Ann. Nuclear Energy*, to appear.
4. E. HORST, "Zur Existenz Globaler Klassischer Lösungen des Anfangswertproblems der Stellardynamik," Disertation, München, 1979.
5. E. HORST, On the existence of global classical solution of stellar dynamics, *in* "Mathematical Problems in the Kinetic Theory of Gases" (D. C. Pack and H. Neunzert, Eds.), Frankfurt, 1980.
6. E. HORST, On the classical solutions of the initial value problem for the unmodified nonlinear Vlasov equation I, *Math. Meth. Appl. Sci.* **3** (1981), 229–248.
7. E. HORST, On the classical solutions of the initial value problem for the unmodified nonlinear Vlasov equation II, *Math. Meth. Appl. Sci.* **4** (1982), 19–32.
8. S. V. IORDANSKII, The Cauchy problem for the kinetic equation of plasma, *Amer. Math. Soc. Transl. Ser. 2* **35** (1964), 351–363.
9. R. KURTH, Das anfangswertproblem der stellar dynamik, *Z. Astrophys.* **30** (1952), 213–229.
10. S. UKAI AND T. OKABE, On classical solution in the large in time of two-dimensional Vlasov's equation, *Osaka J. Math.* **15** (1978), 245–261.

11. S. Wollman, "Classical Solutions to the Vlasov–Poisson System of Equations," Technical Report No. 329, Department of Mathematics, University of New Mexico, 1977.
12. S. Wollman, Global-in-time solutions of the two-dimensional Vlasov–Poisson system, *Comm. Pure Appl. Math.* **33** (1980), 173–197.
13. S. Wollman, The spherically symmetric Vlasov–Poisson system, *J. Differential Equations* **35** (1) (1980), 30–35.